

GROUND STATE OF A SYSTEM OF N HARD CORE QUANTUM PARTICLES IN 1-D BOX

Yatendra S. Jain

Department of Physics, North-Eastern Hill University,
Shillong-793 022, Meghalaya, India

Abstract

The ground state of a system of N impenetrable hard core quantum particles in a 1-D box is analyzed by using a new scheme applied recently to study a similar system of two such particles [*Centl. Eur. J. Phys.*, 2(4), 709 (2004)]. Accordingly, each particle of the system behaves like an independent entity represented by a *macro-orbital*, -a kind of pair waveform identical to that of a pair of particles moving with $(q, -q)$ momenta at their *center of mass* which may have any momentum K in the laboratory frame. It concludes: (i) $\langle A\delta(x) \rangle = 0$, (ii) $\langle x \rangle \geq \lambda/2$ and (iii) $q \geq q_o (= \pi/d)$ (with $d = L/N$ being the average nearest neighbor distance), *etc.* While all bosons in their ground state have $q = q_o$ and $K = 0$, fermions have $q = q_o$ with different K ranging between 0 and $K = K_F$ (the Fermi wave vector). Independent of their bosonic or fermionic nature, all particles in the ground state define a close packed arrangement of their equal size wave packets representing an ordered state in phase (ϕ -)space with $\Delta\phi = 2n\pi$ (with $n = 1, 2, 3, \dots$), $\langle x \rangle = \lambda/2 = d$, and $q = q_o$. As such our approach uses greatly simplified mathematical formulation and renders a visibly clear picture of the low energy states of the systems and its results supplement earlier studies in providing their complete understanding.

Keywords: Wave mechanics, δ -particles, N particles, 1-D box, macro-orbital approach

PACS : 03.65.-w, 03.65.Ca, 03.65.Ge, 03.75.Nt

email: ysjain@email.com

1. Introduction

Wave mechanics of a 1-D system of N identical particles interacting through *hard core* (HC) potential of zero or non-zero range has been a subject of great importance for the last seven decades [1-21]. The development of the subject has been facilitated by the fact (demonstrated by first few studies by Tonks [1], Girardeau [2,3], and Lieb and Liniger [4]) that the problem is exactly solvable. Identical conclusion has been made later [22, 23] for a similar system of particles interacting through inverse square potential. These developments have been elegantly reviewed by Popov [24], Korepin, *et al* [25], Mattis [26] and most recently by sutherland [27]. As evident from the number of such studies [28-44], interest in the subject is renewed recently after the use of *Bose Einstein condensation* (BEC) in dilute gases in experimental realization of many body systems [45-54] that can be described by 1-D Hamiltonian with δ -potential interaction [55, 56]. The scheme of

theoretical analysis of these systems uses Bethe ansatz [57] for N body wave function with bosonic/fermionic symmetry (as the case demands) and suitable (periodic/cyclic) boundary conditions. Proposed initially by Lieb and Liniger [4], properties of such a system can be determined theoretically by introducing a dimensionless parameter, $\gamma = mg_{1D}/n\hbar^2$ (with m , -the mass of a particle, n , -the number density of particles and g_{1D} , -the 1-D coupling constant related to 3-D s -wave scattering length a). One identifies Tonks-Girardeau (TG) regime of strong coupling (large g_{1D}) and low density (n) by $\gamma \approx \infty$, Bogoliubov regime [56] (or decoherent regime [53]) of nearly free bosons by $\gamma \approx 0$, and Lieb and Liniger (LL) regime of weak coupling (low g_{1D}) and high density by intermediate values (*viz.*, $\infty > \gamma > 0$). One also defines Gross-Pitaevskii (GP) regime [53] by $1 > \gamma > 0$. Apparently, the behavior of a system of given g_{1D} , can be studied in its different regimes simply by changing n from 0 to ∞ . One of the important conclusions of these studies reveals that the physics of a bosonic system characterized by $\gamma \approx \infty$ resembles with that of free fermions.

In this paper, we analyze the *ground state* (G-state) of a 1-D system of N HC particles by using a slightly different scheme. The excitations of the system would be investigated in our forthcoming article. Our scheme uses the wave mechanics of a pair of HC particles (*identified as the basic unit of the system*) as its basis; it has been investigated in our recent paper [58] for two HC particles trapped in a 1-D box [58]. Our approach renders exact solutions. In view of the equivalence of impenetrable δ -repulsion with HC interaction of non-zero range [*i.e.*, $V_{HC}(x)$, defined by $V_{HC}(x < \sigma) = \infty$ and $V_{HC}(x \geq \sigma) = 0$ with σ being the HC diameter of a particle] demonstrated by Huang [59] and physically argued in [58], one may find that our results for δ -repulsion can be applied to particles of any σ , particularly, in the low energy states of $\lambda/2 \geq \sigma$.

Our results differ from those of extensively used scheme(s) of others [24-27], particularly, for the low energy states where the well known consequences of wave particle duality, *viz.*, inter-particle phase correlation, zero-point repulsion, quantum size, *etc.* (defined and discussed in Section-3 for their better understanding), become important. Our scheme provides a mathematically simple, unified, and clear picture of 1-D systems. As such one may find that the present study supplements earlier studies in providing a better understanding of our 1-D systems and its basic results resemble with those of : (i) a single particle in 1-D box [60, 61], two HC particles in a similar box ([58], and (iii) long awaited microscopic theory of a 3-D system of interacting bosons [62-64] which explains the properties of liquid 4He (a well known representative of such systems) with unmatched accuracy, simplicity and clarity. This demonstrates the accuracy and usefulness of our scheme. As reported briefly in [65, 66], our scheme provides a theoretical framework that unifies the physics of widely different systems (including low dimensional systems) of interacting bosons and fermions, atomic nucleus, newly discovered BEC state, *etc.* and helps in revealing the basic foundations of the microscopic theory of superconductivity.

The paper has been arranged as follows. While the important aspects of our N -body system are described in Section-2, the important inferences of our study of two HC particles in 1-D box, which serves the basis of this work, are summed up in Section-3.

This is followed by a detailed analysis of the ground state of the system in Section-4 and concluding remarks in Section-5.

2. Important Aspects of N-Body System

The hamiltonian of a 1-D system of N particles interacting through a two body impenetrable δ -potential can be expressed as

$$H(N) = -\frac{\hbar^2}{2m} \sum_i^N \frac{\partial^2}{\partial x_i^2} + \sum_{i>j} A\delta(x_{ij}) \quad (1)$$

where A , representing the strength of Dirac delta function ($\delta(x_{ij})$), is such that $A \rightarrow \infty$ with $x_{ij} \rightarrow 0$, while other notations have usual meaning. Possible interactions such as spin-spin interaction are not included in Eqn. 1 by presuming that these can be treated perturbatively.

Analysing the basic nature of the possible dynamics of two particles (say P1 and P2) in the system, we note that: (i) P1 and P2 encounter δ -potential only when they suffer a collision, (ii) in a two body collision, they simply exchange their momenta, k_1 and k_2 , while in a many body collision, where they also collide simultaneously with other particle(s), they could be identified to jump from their state of k_1 and k_2 (or their relative momentum $k = k_2 - k_1$ and *center of mass* (CM) momentum $K = k_1 + k_2$) to that of new momenta k'_1 and k'_2 (or k' and K'), and (iii) between two collisions each of them has free particle motion. Evidently, a state of two particles even in the presence of other particles of our system can be identified characteristically with the state of only two HC particles in 1-D space. In other words a pair of particles forms the basic unit of our system and this agrees with the fact that particles in the system interact through a pair potential.

3. Dynamics of two HC particles (P1 and P2)

We note that the dynamics of P1 and P2 can be described by the Schrödinger equation for $H(2)$ [Eqn. 1 with $N = 2$] expressed in the CM coordinate system as

$$\left(-\frac{\hbar^2}{4m} \frac{\partial^2}{\partial X^2} - \frac{\hbar^2}{m} \frac{\partial^2}{\partial x^2} + A\delta(x) \right) \Psi(x, X) = E\Psi(x, X), \quad (2)$$

with

$$\Psi(x, X) = \psi_k(x) \exp[i(KX)] \quad (3)$$

where $\psi_k(x)$ and $\exp[i(KX)]$, respectively, define the relative and CM motions of P1 and P2. We have

$$x = x_2 - x_1 \quad \text{and} \quad k = k_2 - k_1 = 2q, \quad (4)$$

$$X = (x_1 + x_2)/2 \quad \text{and} \quad K = k_1 + k_2 \quad (5)$$

with x and k , respectively, describing the relative position and relative momentum of P1 and P2, while X and K , similarly, defining the position and momentum of their CM. Without loss of generality, one may also define

$$k_1 = -q + \frac{K}{2} \quad \text{and} \quad k_2 = q + \frac{K}{2}. \quad (6)$$

It is evident that $\psi_k(x)$ is a solution of

$$\left(-\frac{\hbar^2}{m} \frac{\partial^2}{\partial x^2} + A\delta(x)\right) \psi_k(x) = E_k \psi_k(x) \quad (7)$$

where $E_k = E - \hbar^2 K^2 / 4m$. With a view to find $\psi_k(x)$, we note that P1 and P2 experience zero interaction at $x \neq 0$ where each of them can be described by separate plane waves such as $u_{k_i}(x_i, t) = \exp(ik_i x_i) \exp(-iE_i t/\hbar)$ (assumed to have unit normalization), however, for the possible superposition of these waves, their quantum state is better expressed by

$$\Psi(x_1, x_2, t)^\pm = 1/\sqrt{2} [u_{k_1}(x_1, t) u_{k_2}(x_2, t) \pm u_{k_2}(x_1, t) u_{k_1}(x_2, t)]. \quad (8)$$

Since the state function of two impenetrable HC particles must vanish at $x_1 = x_2$, $\Psi(x_1, x_2, t)^+$ (having *ve* symmetry for their exchange) does not represent the desired function, while $\Psi(x_1, x_2, t)^-$ of *-ve* symmetry has no such problem. We addressed this problem in our recent study [58] of the wave mechanics of two impenetrable HC particles in 1-D box and used another method to obtain the right wave function of *ve* symmetry. Evidently, using our results of [58], we express a state of two HC fermions/bosons (in the CM coordinates) by

$$\zeta(x, X, t)^\pm = \zeta_k(x, t)^\pm \exp[i(KX)] \exp[-i(E_k + E_K)t/\hbar] \quad (9)$$

with

$$\zeta_k(x, t)^- = \sqrt{2} \sin(kx/2) \exp[-iE_k t/\hbar] \quad (10)$$

of fermionic symmetry and

$$\zeta_k(x, t)^+ = \sqrt{2} \sin(|kx|/2) \exp[-iE_k t/\hbar] \quad (11)$$

of bosonic symmetry. Note that $\zeta(x, X, t)^-$ is a form of $\Psi(x_1, x_2, t)^-$ (Eqn. 8) expressed in CM coordinates. Although, two impenetrable HC particles in 1-D are expected to retain the order of their locations (*e.g.* $0 < x_1 < x_2 < L$ for two particles in a 1-D box with infinite potential walls located at $x = 0$ and $x = L$) but Eqns. 9-11 remain valid for describing the state of P1 and P2 since the two particles during their collision do exchange k_1 and k_2 which is equivalent to exchanging sides because the particles are identical. In fact in a state of wave mechanical superposition of two identical particles, one has no means to ascertain whether the particles exchanged their momenta or their positions and we may, justifiably, analyze Eqns. 9-11 to find important aspects of the wave mechanics two HC particles as follows.

(i) *SMW state* : $\zeta(x, X, t)^\pm$ represents a kind of *standing matter wave* (SMW) which modulates the probability, $|\zeta(x, X, t)^\pm|^2 = |\zeta_k(x, t)^\pm|^2$, of finding the two HC bosons (or fermions) at their relative phase position $\phi = kx$ in ϕ -space. However, the equality, $|\zeta_k(x, t)^-|^2 = |\zeta_k(x, t)^+|^2$, renders an important fact that the relative configuration of P1 and P2 in $\zeta_k(x, t)^\pm$ states is independent of their fermionic or bosonic symmetry. Although, t -dependent terms in Eqns. 9-11 have their own importance and they help in

having a better understanding of the SMW nature of $\zeta(x, X, t)^\pm$, nevertheless these can be dropped for their no impact on our impending analysis.

(ii) *Symmetry of relative motion* : The relative motion of P1 and P2 maintains a center of symmetry at their CM and this implies that

$$x_{CM}(1) = -x_{CM}(2) = x/2 \quad \text{and} \quad k_{CM}(1) = -k_{CM}(2) = q \quad (12)$$

where x_{CM} and k_{CM} , respectively, represent the position and momentum of a particle with reference to the CM of P1 and P2. Evidently, two particles in $\zeta_k(x, t)^\pm$ state have equal and opposite momenta ($q, -q$) with respect to their CM whose motion in the laboratory frame can be defined by a plane wave ($\exp(iKX)$) of momentum K and this agrees with what we learn from Eqn. 6.

(iii) *MS and SS states* : We note that $\zeta(x, X, t)^\pm$ resulting from the superposition of two plane waves of k_1 and k_2 could be identified to represent the *mutual superposition* (MS) of P1 and P2 because $\zeta(x, X, t)^\pm$ is, basically, an eigenstate of the energy operators of the relative and CM motions of two particles rather than of individual particle. However, it could also be identified as a state of the *self superposition* (SS) of individual particle (P1 or P2) by identifying $\zeta(x, X, t)^\pm$ as a *macro-orbital* (cf. Point-vii below) on the basis of the following analysis. To understand the meaning of self superposition one may track down the motions of P1 and P2 separately and find that each of them (say P1), after its collision with P2, has superposition of its pre-collision plane wave $u_{k_1}(x_1, t)$ with its post-collision plane wave $u_{k'_1}(x_1, t)$ (with its new momentum $k'_1 = k_2$ because during the collision P1 exchanges its momentum with P2 and *vice versa*); similar observation applies to P2. As such $\zeta_k(x, t)^\pm$ part of our SS state is not different from the state of a particle trapped in 1-D box. However, since P1 and P2 are identical particles we have no means to identify whether the two particles have their mutual superposition or a self superposition of individual particle. Consequently, $\zeta(x, X, t)^\pm$ could be used identically to represent either the MS state of P1 and P2 or the SS state of either particle. It may be mentioned that two plane waves of two particles can, in principle, have their superposition independent of their relative position x and relative momentum k , however, as shown by experiments the wave nature of particles seem to influence the behavior of a system of particles only when their $\lambda = 2\pi/q$ compares with their x which implies that an effective superposition of two particles leading to their MS/SS state exists only in low energy state with $\lambda \approx x$.

(iv) *Characteristics of relative motion* : $\zeta_k(x, t)^\pm$ has a series of antinodes of size $\lambda/2$ between different nodes at $x = \pm n\lambda/2$ ($n = 0, 1, 2, 3, \dots$) where $\lambda = 2\pi/q$ represents the de Broglie wave length related to the relative motion of P1 and P2. Evidently, two particles can be confined to the *shortest possible space* of λ size without disturbing their $\zeta_k(x, t)^\pm$ state (say, by placing two impenetrable potential walls at the nodal points, $x = \pm\lambda/2$ and for such a confinement we have [58] $\langle x \rangle = I'/I = \lambda/2$ with integrals $I' = \langle \zeta_k(x, t)^\pm | x | \zeta_k(x, t)^\pm \rangle$ and $I = \langle \zeta_k(x, t)^\pm | \zeta_k(x, t)^\pm \rangle$ performed between $x = 0$ (the minimum possible x) to $x = \lambda$ (the maximum possible x). Evaluating similar results

for $\langle \phi \rangle$ (as shown in [58]), we find that a state two HC particles in free space is characterized by

$$\langle x \rangle \geq \lambda/2, \quad \text{and} \quad \langle \phi \rangle \geq 2\pi. \quad (13)$$

This implies that any experiment, that keeps track of the relative dynamics of two HC particles, would find that P1 and P2 never reach closer than $\langle x \rangle_o = \lambda/2$ (shortest possible $\langle x \rangle$) and $\langle \phi \rangle_o = 2\pi$ (shortest possible $\langle \phi \rangle$). Similarly, Eqns. 12 and 13 reveal that the position of P1 and P2 as seen from their CM can not be closer than $\langle x_{CM}(1) \rangle_o = -\lambda/4$ and $\langle x_{CM}(2) \rangle_o = \lambda/4$ or *vice versa*. Finally, we also find that

$$\langle A\delta(x) \rangle = |\zeta_k(x)|_{x=0}^2 = 0, \quad (14)$$

which has been analyzed for its general validity in [58] (see Appendix-A of arXiv.org/quant-ph/0603233) which clearly shows that $\langle A\delta(x) \rangle = 0$ is valid for all physically relevant situations rendering

$$\langle H(2) \rangle = (\hbar^2/4m)(K^2 + k^2) = (\hbar^2/2m)(k_1^2 + k_2^2). \quad (15)$$

(v) *Quantum size of a particle* : In view of Eqn. 13, it is evident that two particles by themselves do not assume a state of $\langle x \rangle < \lambda/2$ (with $\lambda = 2\pi/q$) which indicates that each of them exclusively occupies $\lambda/2$ space; being identical particles with equal $|q|$, they are expected to share λ space equally. Hence as used in this paper we identify $\lambda/2$ as the effective size of a particle particularly for low energy states of $q \leq \pi/\sigma$ and name it as *quantum size*. For a better understanding of this meaning, one may consider the wave associated with P1 as a probe to scale the size of P2 or *vice versa* and apply the principle of image resolution to find that P1 (or P2) would be able to resolve the σ size of P2 (or P1) only if $\lambda/2 \leq \sigma$ (*i.e.* $q \geq \pi/\sigma$) implying that in such a case they see each other as particles of size σ . However, in case $\lambda/2 > \sigma$, they can not resolve σ and would see each other as the objects of size $\lambda/2$ limited by their capacity to resolve. As such the effective size of P1 and P2 is a q -dependent quantity for $\lambda/2 > \sigma$ (or $q < \pi/\sigma$) and q -independent quantity for $\lambda/2 \leq \sigma$ (or $q \geq \pi/\sigma$). Evidently, the use of quantum size for $\lambda/2$ is justified to distinguish the two situations. This meaning seems to qualitatively agree with the meaning of “*quantum spread of a particle*” as used by Huang [59]. However, on quantitative scale it seems to represent the minimum possible quantum spread (*limited by the uncertainty principle*) of a particle of momentum q but it, evidently, differs from the meaning of “quantum size” as used in relation to the *quantum size effects* on the properties of thin films and small clusters of atoms [67]. The fact that a particle in its ground state in a box has a spread of $\lambda/2 = d$ (*the size of the box*) also indicates that the particle exclusively occupies $\lambda/2$ space because any attempt to decrease this space size (say by reducing d) would push the particle to have new momentum/energy but once again the space occupied would be a new $\lambda/2$.

(vi) *Zero point repulsion* : Since two HC particles do not assume a state of $\langle x \rangle < \lambda/2$ (*cf.*, Eqn. 13), one would obviously like to identify the responsible force. To this effect we note that $\langle x \rangle < \lambda/2$ represents an overlap two particles of effective size $\lambda/2$ and

this implies that particles in this state have higher energy in comparison to the state of $\langle x \rangle \geq \lambda/2$. Evidently, the two particles in the state of $\langle x \rangle < \lambda/2$ experience a kind of mutual repulsion (or *the zero point repulsion*) unless they assume a state of $\langle x \rangle \geq \lambda/2$. Thus $\lambda/2$ represents the effective size of a HC particle of $q < \pi/\sigma$ and the range of zero-point repulsion.

(vii) *Macro-orbital representation of a particle* : Although, two particles in $\zeta(x, X)^\pm$ states can be identified to have inter-particle phase correlation ($g(\phi) = |\zeta(x, X)^\pm|^2 = |\sin(\phi/2)|^2$) which can keep them locked at $\langle \phi \rangle = 2n\pi$ ($n = 1, 2, \dots$) in ϕ -space, nevertheless they do not form a bound pair in x -space because they either experience mutual repulsion when $\langle x \rangle < \lambda/2$ or no force when $\langle x \rangle \geq \lambda/2$. Evidently, each of them in $\zeta(x, X)^\pm$ state represents an independent entity described by a separate pair waveform, say, $\xi(x_{(i)}, X_{(i)}) \equiv \zeta(x, X)$; the subscript (i) refers to i -th particle. To distinguish $\xi(x_{(i)}, X_{(i)})$ from $\zeta(x, X)$, we propose to call the former a macro-orbital [58] and obtain the same by replacing x , X , k and K in $\zeta(x, X)$ by $x_{(i)}$, $X_{(i)}$, $k_{(i)}$ and $K_{(i)}$, respectively. This renders

$$\xi(x_{(i)}, X_{(i)}) = B\zeta_{q_{(i)}}(x_{(i)}) \exp[i(K_{(i)}X_{(i)})] \quad (15)$$

where B is the normalization constant and $\zeta_{q_{(i)}}(x_{(i)})$ is that part of a macro-orbital which does not overlap with similar part of other macro-orbital. In view of what has been argued in point (iii) above, the macro-orbital representation is consistent with MS/SS state of P1 and P2.

(viii) *Macro-orbital as an eigenfunction* : A macro-orbital ($\xi(x_{(i)}, X_{(i)})$) is a derived form of a wavefunction which identifies a particle to have two different motions (*cf.* Eqn. 15) : the q -motion of energy $E(q_{(i)}) = \hbar^2 q_{(i)}^2 / 2m = \hbar^2 k_{(i)}^2 / 8m$ which decides its quantum size $\lambda_{(i)}/2 = \pi/q_{(i)}$ and K -motion of energy $E(K_{(i)}) = \hbar^2 K_{(i)}^2 / 8m$ which represents a kind of free motion of the particle. Evidently, this form does not fit, as a solution, with the form of Schrödinger equation (Eqn. 2). However, the corresponding hamiltonian can be rearranged to obtain Eqn. 2 in a form with which $\xi(x_{(i)}, X_{(i)})$ is compatible and to this end we define

$$h_i = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2}, \quad h(i) = \frac{h_i + h_{i+1}}{2} \quad \text{and} \quad H(2) = h(1) + h(2) \quad (16)$$

with $i = 1$ or 2 for a system of $N = 2$, $h_{N+1} = h_1$, h_i being the kinetic energy operator of i -th particle in unpaired format of P1 and P2, and $h(i)$ is the same in their paired format. We have

$$h(i) = -\frac{\hbar^2}{8m} \frac{\partial^2}{\partial X_{(i)}^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_{(i)}^2} \quad (17)$$

which is such that

$$h(i)\xi(x_{(i)}, X_{(i)}) = [(E(q)_{(i)} + E(K)_{(i)})/2]\xi(x_{(i)}, X_{(i)}). \quad (18)$$

4. N-Particle System and Macro-orbital Representation

(i). *Rearrangement of $H(N)$* : For the consistency of macro-orbital representation, we use Eqn. 16 to rearrange

$$H(N) = \sum_i^N h_i + \sum_{i>j} A\delta(x_{ij}) \quad (19)$$

as

$$H(N) = \sum_i^N h(i) + \sum_{i>j} A\delta(x_{ij}) \quad (20)$$

or

$$H(N) = \sum_i^N \left[-\frac{\hbar^2}{8m} \frac{\partial^2}{\partial_{X(i)}^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial_{x(i)}^2} \right] + \sum_{i>j} A\delta(x_{ij}). \quad (21)$$

(ii). *State Functions* : Using N macro-orbitals for N particles, we construct a state function of the system by following standard procedure. Assuming that Ψ_n represents n -th state of energy E_n , we have

$$\Psi_n = \phi_n(q)\phi_n(K), \quad (22)$$

with

$$\phi_n(q) = \Pi_{i=1}^N (2/L)^{N/2} \sin [q_{(i)}x_{(i)}], \quad (23)$$

and

$$\phi_n(K) = B \sum_P (\pm 1)^P \Pi_{i=1}^N [\exp ({}_PK_{(i)}X_{(i)})]. \quad (24)$$

Since each particle occupies a cage between its neighbors, $q_{(i)}$ in Eqn. 23 should be integer multiple of $\pi/d_{(i)}$ ($d_{(i)}$ = size of the cage); to a good approximation, $q_{(i)}$ can also be assumed to be an integer multiple of π/L if L is macroscopically large. In Eqn. 24, we have $B = (N!L^N)^{-1/2}$ as the normalization constant, $K_{(i)} = N\pi/L$ (with $N = 1, 2, 3, \dots$) and $\sum_P (\pm 1)^P$ as the sum of $N!$ products of N plane waves obtained by permuting different $K_{(i)}$ over N particles with P being the number of performed permutations. While the use of $(+1)^P$ or $(-1)^P$ depends on the bosonic or fermionic nature of particles, the condition $q_{(i)} \geq \pi/d_{(i)}$ follows from the HC nature of particles (bosons and fermions alike). Since different particles can, in principle, have different $q_{(i)}$, their permutation over N particles renders $N!$ different $\phi_n(q)$ and, therefore, similar number of Ψ_n . Consequently, the general form of a state function that should reveal the physics of the system should be expressed by

$$\Phi_n = (N!)^{-1/2} \sum_{j=1}^{N!} \Psi_n^{(j)}. \quad (25)$$

(iii). *Energy Eigenvalues* : The energy eigenvalue E_n can be expressed by

$$E_n = \frac{\langle \Phi_n | H_o(N) + \sum_{i<j} A\delta(x_{ij}) | \Phi_n \rangle}{\langle \Phi_n | \Phi_n \rangle} = \frac{\langle \Phi_n | \sum_i^N h(i) | \Phi_n \rangle}{\langle \Phi_n | \Phi_n \rangle} \quad (26)$$

which renders

$$E_n = \sum_i^N \left[\frac{\hbar^2 K_{(i)}^2}{8m} + \frac{\hbar^2 q_{(i)}^2}{2m} \right] \quad (27)$$

since $\langle A\delta(x_{ij}) \rangle$ vanishes (*cf.*, Eqn. 14) for every pair of HC particles. One can also conclude $\langle A\delta(x_{ij}) \rangle = 0$ by using N body wave function Φ_n and the detailed derivations, as used to conclude similar result for a 3-D N body system [63].

(iv). *G-state Energy*: While $K_{(i)}$ has zero value for all bosons in their G-state, its different values ranging between 0 and K_F (Fermi wave vector) in the G-state of a fermionic system follow Fermi-distribution. To fix possible values of $q_{(i)}$ for which E_o is minimum, we note that each particle satisfies $\lambda_{(i)}/2 \leq d_{(i)}$ which implies that $\lambda_{(i)}/2$ space exclusively belongs to a HC particle of momentum $q_{(i)}$; this agrees with the volume excluded condition envisaged by Kleban [68] for HC particles in 3-D systems. Since each particle in the G-state has lowest energy or largest possible $\lambda_{(i)}/2$, the net energy of q -motion can be expressed as

$$E_o = \sum_i^N \frac{h^2}{8md_{(i)}^2} \quad \text{with} \quad \sum_i^N d_{(i)} = L \quad (\text{constant}) \quad (28)$$

by assuming that particles occupy cages of different size $d_{(i)}$. Simple algebra reveals that E_o has its minimum value for $d_{(1)} = d_{(2)} = d_{(3)} \dots = d_{(N)} = d$. Obviously, for a bosonic system we have

$$E_o^{bose} = N \frac{h^2}{8md^2} = N\varepsilon_o. \quad (29)$$

which does not differ from similar result for 3-D bosonic system [64]. Accounting for an additional contribution of K -motions to the G-state energy of a fermionic system, we have

$$E_o^{fermi} = N\varepsilon_o + \frac{1}{4}\epsilon_K, \quad (30)$$

where ϵ_K is the total energy of K -motions at $T = 0$ in a system of non-interacting fermions. The factor $(\frac{1}{4})$ accounts for the fact that each particle of our system in its macro-orbital representation serves as an entity of mass $4m$ for its K motion. For a 3-D system, we use the well known relations [70] for the Fermi level energy $E_F \approx h^2/8md^2 = \varepsilon_o$ and $\epsilon_K = \frac{3}{5}NE_F$ and obtain

$$E_o^{fermi} \approx N\varepsilon_o + \frac{3}{20}N\varepsilon_o \approx 1.15N\varepsilon_o. \quad (31)$$

However, for 1-D system, we find $\epsilon_K \approx \frac{1}{6}NE_F$ and $E_F = \frac{1}{4}\varepsilon_o$ to get

$$E_o^{fermi} = N\varepsilon_o + \frac{1}{24}NE_F \approx N\varepsilon_o + \frac{1}{96}N\varepsilon_o \approx 1.01N\varepsilon_o. \quad (32)$$

From Eqns. 31 and 32, it is clear that E_o^{fermi} (both for 3-D system and 1-D system) is only marginally higher than E_o^{bose} (Eqn. 29) which indicates that K -motions have very little contribution to the G-state of a fermionic system ($\approx 15\%$ in case of 3-D system, *cf.* Eqn. 31 and $\approx 1\%$ in case of 1-D system, *cf.* Eqn. 32).

(v). *Process of Reaching G-State*: In what follows from the above discussion the free energy (F) of the system can be identified to be a sum of two contributions $F(K)$ and $F(q)$, respectively, arising from K - and q - motions of particles. While $F(K)$ refers to a gas of non-interacting quantum quasi-particles of mass $4m$ constrained to move in one direction only, $F(q)$ could be equated, to a good approximation, to $N\varepsilon_o$ as shown for the electron fluid (a fermionic system) in a conductor [66] as well as for liquid ^4He (a bosonic system) [64]. As concluded by Mermin and Wagner [71], we note that $F(K)$ is not expected to show any transition in case of a 1-D system, while $F(q) = N\varepsilon_o$ does not depend explicitly on T and P . Consequently, there is no means to find a relation for a T and P for any possible transition in our system except by using possible physical arguments. In this context, we find that for a given particle density of the system, $(d - \lambda/2)$ decreases with decreasing T and at certain $T = T_c$, when $(d - \lambda/2)$ (with $d = L/N$) vanishes at large, q motions get frozen onto zero point motions of $q = q_o = \pi/d$ and the system moves from a state of $\lambda/2 \leq d$ to that of $\lambda/2 = d$. While $\lambda/2 \leq d$ corresponds to $\phi(= 2qd) \geq 2\pi$ which represents randomness of ϕ positions and, therefore, a disordered state in ϕ -space, $\lambda/2 = d$ defines a state of $\phi = 2\pi$, - an ordered state in ϕ -space. Thus the system, on its cooling through T_c , moves from its disordered state to ordered state of its particles in the ϕ - space but this change does not represent an onset of a finite T phase transition. This is because the change refers to the relative configuration of particles (relative position ($x = d$), relative momentum ($k = 2q$) and relative phase position ($\phi = 2qd$)) which remains unaltered with fall in T from $T = T_c$ to $T = 0$. Evidently, the said change at $T = T_c$ leads all the N particles to assume a configuration of the ground state of their q -motions which naturally have no thermal energy at all $T \leq T_c$. In other words, the relative configuration of the system is effectively at $T = 0$ and the thermal energy of the system at all $T \leq T_c$ comes entirely from the K -motions only. This clearly means that the said order-disorder of particles is effectively a $T = 0$ change which does not violate Mermin-Wagner theorem [71] forbidding a transition in 1-D and 2-D systems at finite T .

(vi). *Merger of $N!$ micro-states* : We note that with all $q_{(i)} = q_o$, different $\Psi_n^{(j)}$ of Φ_n (Eqn. 25) become identical and the latter attains the form of a single Ψ_n (Eqn. 22) which implies that all the $N!$ microstates merge into one at $T = T_c$ and the entire system attains a kind of oneness, as envisaged by Toubes [72]; the system at $T \leq T_c$ is therefore described by

$$\Phi_n(S) = \phi_n^o(q_o)\phi_n^e(K), \quad (33)$$

where $\phi_n^o(q_o)$ and $\phi_n^e(K)$, respectively, define the relative configuration of the ground state and the collective excitations of the system.

(vii). *Quantum Correlation Potential* : The inter-particle *quantum correlation potential* (QCP) originating from the wave nature of the particles can be obtained by comparing the partition function (under quantum limit of the system), $Z_q = \sum_n \exp(-E_n/k_B T) |\Phi_n(S)|^2$ and its classical equivalent, $Z_c = \sum_n \exp(-E_n/k_B T) \exp(-U_n/k_B T)$. Here $\Phi_n(S)$ is given by Eqn. 33. This can be justified because $\Phi_n(S)$ is basically nothing but a product of paired wave functions obtained by simple superposition of plane waves and our conclusion that two particles in their SMW configuration satisfy $d \geq \lambda/2$ screens out the HC interaction. Simplifying U_n one easily finds that the pairwise QCP has two components.

A U_{ij}^s , pertaining to the q -motion of particles, controls the ϕ -position of a particle and we have

$$U_{ij}^s = -k_B T_o \ln[2 \sin^2(k_{ij} x_{ij}/2)], \quad (34)$$

with $k_{ij} = k_i - k_j$ and $x_{ij} = x_i - x_j$ and T replaced by T_o because T equivalent of q motion energy ε_o at all $T \leq T_c$ is T_o . U_{ij}^s has its minimum value ($-k_B T_o \ln 2$) at $\phi = (2n + 1)\pi$ and maximum value (∞) occurring periodically at $\Delta\phi = 2n\pi$ (with $n = 1, 2, 3, \dots$). We note that U_{ij}^s increases by

$$\frac{1}{2}C(\delta\phi)^2 = \frac{1}{4}k_B T_o (\delta\phi)^2 \quad (35)$$

for any small change ($\delta\phi$) in ϕ around $\phi = (2n + 1)\pi$ where it has its minimum value ($-k_B T_o \ln 2$). This indicates that particles experience a force $= -C\delta\phi$ (force constant $C = \frac{1}{2}k_B T_o$) which is naturally responsible for $\delta\phi = 0$ and an ordered state in ϕ -space. However, since U_{ij}^s is not the real interaction that may manipulate d , the state of order is achieved by driving all q towards q_o .

The second component of QCP, pertains to plane wave K motions, and it can be expressed by

$$U_{ij} = -k_B T \ln \left[1 \pm \exp \left(-\frac{2\pi |X_2 - X_1|^2}{\lambda_T^2} \right) \right], \quad (36)$$

by following standard procedure [73 and 74] applied to plane wave motion of non-interacting particles; while $\lambda_T = h/\sqrt{2\pi(4m)k_B T}$ representing the thermal de Broglie wave length is associated with K -motions (a kind of free motion where each particle appears to have $4m$ mass), $+ve(-ve)$ sign stands for a bosonic(fermionic) system. In case of a bosonic system, U_{ij} has its minimum ($= -k_B T \ln 2$) at $|X_2 - X_1| = 0$ implying that U_{ij} facilitates bosons to occupy a common X point. However, U_{ij} for a fermionic system has its maximum ($= \infty$) at $X_2 = X_1$ and, therefore, forbids two fermions from occupying a common X .

(viii). *Negative Thermal Expansion* : In what follows from Sections-4(vii), the system on its cooling moves from its dis-ordered state in ϕ -space at $T > T_c$ to an ordered state at $T \leq T_c$. The operational force for this change lies with : (i) QCP [U_{ij}^s , Eqn. 34] which modulates the ϕ -positions of particles at $\Delta\phi = 2n\pi$, and (ii) the zero point repulsion (*cf.* Section-3.(vi)) which keeps two particles at $< x > \geq \lambda/2$. Consequently, when $\lambda/2$ increases with falling T , each particle pushes its neighbors to make space for its increased quantum size ($\lambda/2$). However, the increase in $\lambda/2$ virtually stops at T_c when q -motions of particles get frozen at $q = q_o$ (or $\lambda/2 = d$) because the box gets totally occupied by closely packed wave packets of N particles with $L = N\lambda/2$. Evidently the zero-point repulsion of a particle on its nearest neighbor, at this stage, renders a force which tries to increase the box size, L . In all practical situations where forces restoring L are not infinitely strong, a non-zero increase ($+\delta L$) in L leading to a $-ve$ thermal expansion coefficient of the system is expected around T_c . The experimental observation of $-ve$ thermal expansion co-efficient would, obviously, provide proof for our predictions of : (i) the freezing of q -motions at q_o and (ii) the onset of an order in the ϕ -positions of particles. In this context, it may be noted that in agreement with our similar prediction for a 3-D system,

liquids ^4He and ^3He are really found to have -ve thermal expansion coefficient around 2.17K and 0.55K, respectively [75].

(ix). *Estimation of T_c* : We note that the potential energy contribution to net energy of the system is zero, since $\langle A\delta(x_{ij}) \rangle$ vanishes for all pairs of particles. Even zero-point energy $\varepsilon_o = h^2/8md^2$, seemingly having potential energy character as indicated by its d dependence, represents the energy of residual q -motion which implies that the net energy of the system is kinetic. Consequently, to a good approximation, the lower bound of T_c could be equated to T_{GSE} (the T equivalent of the G-state energy of a particle) with $T_{GSE} \equiv E_o^{bose}/N$ (Eqn. 29) for the bosonic system and $T_{GSE} \equiv E_o^{fermi}/N$ (Eqn. 30) for a fermionic system. However, to obtain more accurate T_c we need to account for $T_{ex} \equiv E_{ex}$, -the energy of thermal excitations (representing the correlated K -motions of particles) present in the system at T_{GSE} and this renders

$$T_c \approx T_{GSE} + T_{ex}. \quad (37)$$

To estimate T_{ex} for a bosonic system we note that the freezing of q -motions at $q = q_o$, immediately follows an onset of the condensation of particles in $K = 0$ state because at this stage the system only has K -motions that may lose energy with falling T . Although, as concluded in Section-3(v), the change at T_c is an effectively $T = 0$ transition but one also finds that a quasi 1-D system realized in different laboratories basically represents a 3-D system where two dimensions are mechanically reduced to the order of inter-atomic separation by increasing two corresponding components of momentum of confined particles and this agrees with the fact that these quasi 1-D systems do exhibit BEC in a manner identical to 3-D systems. Evidently, the $K = 0$ condensation in our quasi 1-D laboratory systems can be identified as the BEC of non-interacting bosons because particles for their K -motions are represented by plane waves. Since the particles for the K -motions appear to have $4m$ mass, we can replace T_{ex} by $\frac{1}{4}T_{BEC}$ to obtain

$$T_c^{bose} = T_o + \frac{1}{4}T_{BEC} = \frac{h^2}{8\pi mk_B} \left[\frac{1}{d^2} + \left(\frac{N}{2.61V} \right)^{2/3} \right] \quad (38)$$

for a bosonic system, where particles are free to move within its volume V on a surface of a constant potential. Similarly, for a system of bosons confined to a harmonic trap we have

$$T_c^{bose} = T_o + \frac{1}{4}T_{BEC} \approx 0.55\hbar\omega N^{1/3}/k_B \quad (39)$$

where we use our result $T_o = \hbar\omega N^{1/3}/(\pi k_B)$ [65] as well as $T_{BEC} = \hbar\omega(N/1.202)^{1/3}/k_B$ obtained for such systems by Groot *et. al.* [76].

The T_c^{bose} of a hypothetical 1-D system of HC bosons may be equated to T_o as its lower bound and to T_c^{bose} value given by Eqn. 38/39 as its upper bound since the K -motion energy in such a 1-D system is expected to be lower than that present in a 3-D system. This indicates that T_o which represents about 66% of T_c^{bose} in Eqn. 38 and about 60% in Eqn. 39 would be more close to T_c^{bose} of a hypothetical 1-D system.

In order to find T_c^{fermi} (the T_c for a Fermi system), we similarly use Eqn. 30 to find $T_{GSE}(\equiv E_o^{fermi}/N)$ as its lower bound and assess E_{ex} at T_{GSE} to find equivalent T_{ex} .

We note that such E_{ex} should be a small fraction of E_o^{fermi} because at such a low T not many fermions are in excited state and T_c^{fermi} can be placed close to $T_{GSE}^+ \approx T_o^+$ (a T slightly above T_o). It is important to note that $T_c^{fermi} = T_o^+$ represents a point at which (i) q -motions get frozen at $q = q_o$, (ii) particles define an ordered state in the phase space and (iii) the system exhibits *-ve thermal expansion coefficient*. This should not be confused with a point below which a Fermi system transforms into a quantum fluid. As analyzed and discussed in [65, 66], the superfluid transition point in a Fermi system falls much below T_o . This is because the fermions always have non-zero K and according to Pauli exclusion principle two fermions with equal q have to have unequal K or *vice versa*. Evidently, when they have equal K , they have unequal q and this possibility does not allow the relative configuration to stabilize with $\Delta\phi = 2n\pi$ unless the thermal energy falls below their binding energy which can be there only if the particles have inherent or induced inter-particle attraction [66].

5. Concluding Remarks

(i). The paper uses a new scheme to analyze the G-state of a 1-D system of N HC quantum particles. It concludes that each particle in the system should better be described by a macro-orbital (*a kind of pair wave form*) since each particle represents a part of the pair of particles (*the basic unit of the system*) having equal and opposite momenta ($q, -q$) at their CM which moves with momentum K in the laboratory frame. This agrees with the fact that particles in the system interact through a two body potential and their individual momenta (k_i with $i = 1, 2, 3, \dots, N$) do not define good quantum numbers as soon as the particles have their wave mechanical superposition. Moreover the fact, that a particle in its macro-orbital representation has two motions [*viz.*, the q - and K - motions] of the pair, -it represents, is consistent with Eqn. 6.

(ii). While all particles in the G-state of both systems (bosonic as well as fermionic) retain q -motions of identically equal $q = q_o = \pi/d$, they all have $K = 0$ for bosonic system and follow Fermi-distribution to occupy different K -states ($K = n\pi/L$ with $n = 1, 2, 3, \dots$) with K ranging from $\pi/L \approx 0$ to K_F in case of a fermionic system.

(iii). Particles in the G-state of both systems define a close packed arrangement of their representative wave packets (each of size $\lambda/2 = d$) with inter-particle phase separation $\Delta\phi = 2n\pi$ ($n = 1, 2, 3, \dots$), nearest neighbor distance $\langle x \rangle = d$ and this packing does not allow them to have relative motion. Consequently, the G-state is a state of collisionless motion which can be identified as the residual zero-point q -motion. It differs from the higher energy states where particles can be perceived to have collisions because their effective size ($\lambda/2$) is smaller than the average space (d) available to each particle.

(iv). While a bosonic system is expected to exhibit *-ve* thermal expansion coefficient at $T \approx T_c^{bose}$, a fermionic system should show such effect around $T_c^{fermi} \approx T_o$ (*cf.* Section-4(viii)). This agrees with similar conclusion for a single particle in 1-D box [61] and two HC particles in similar box [58]. Our scheme applied to 3-D systems [63-66] also predicts similar expansion whose accuracy is established by experimentally observed expansion of systems like liquids 4He and 3He [75].

(v). The G-state configuration of our system (*cf.* point (ii), above) does not assume stability against any perturbation (*viz.*, the flow of the system) because the particles in this configuration are in a state of persisting zero-point repulsion and the necessary inter-particle attraction to counter this repulsion is absent. If such an attraction exists and the system remains fluid even at $T \leq T_c$, then the system can, in principle, exhibit phenomenon like superfluidity below certain T (say T_λ); our analysis of 3-D systems such as liquid ^4He [64] and liquid ^3He [65, 66] concludes that T_λ for a bosonic system falls close to T_o but for a fermionic system it is expected to be orders of magnitude lower than T_o because K -motion energy retained by fermions due to Pauli exclusion is good enough to de-stabilise the G-state configuration unless $k_B T$ falls below per particle binding energy. In our forthcoming paper, we plan to study different aspects of our 1-D system when weak interparticle attraction is added as a perturbation. However, our detailed study of a 3-D bosonic system [64] and a brief qualitative analysis of widely different many body systems (including low dimensional systems and fermionic systems) [65] clearly reveals that a 1-D system realised in a laboratory must have basic properties such as superfluidity and related behavior of 3-D systems if weak inter-particle attraction is present. As shown in Section-4(v) the predicted order-disorder transformation of our 1-D systems is consistent with Mermin Wagner theorem [71]. In this context, it may be emphasized that the transformation is basically a quantum transition expected to occur at $T = 0$; however, it occurs at a non-zero T for the proximity of zero-point q -motions ($q = q_o$) with K -motions which solely represent the thermal motions of particles at $T \leq T_o$.

(vi). The formation of a SMW from the superposition of two plane waves of two HC particles is as natural as the phenomena of interference and diffraction of particles such as strongly interacting electrons, neutrons, helium atoms, *etc.* [77, 78]. Since the nature of interference and diffraction patterns of these strongly interacting particles does not differ from the nature of such patterns for non-interacting photons, it is evident that only wave nature (not the inter-particle interactions) modulates the relative phase positions of particles in their wave superposition. Evidently, the fact that these experiments support the formation of a SMW can not be doubted.

(vii). We understand that, in principle, two particles described by plane waves should have their superposition independent of their d and λ . However the experimental fact, that wave nature dominates the behavior of a many body system only when $\lambda \geq d$ [59, 69], indicates that *effective wave superposition* of two particles becomes possible only when $\lambda \geq d$ and this could be so because no particle in nature manifests a real plane wave. Since the important consequences of wave superposition (*cf.* Section-3.0) such as zero point repulsion leading to $\langle x \rangle \geq \lambda/2$, inter-particle ϕ -correlation which arranges particles at $\Delta\phi = 2n\pi$ and renders coherence of particle motion, *etc.* (incorporated in the present analysis) are not used in [1-6], our results are expected to differ from those of [1-6]. Guided by this observation and the experimental support to the formation of a SMW (Point-vi, above), we may highlight the said differences by comparing our results with those of [4] for $\gamma = \infty$ case. If our inference for allowed $q = q_o = \pi/d$ and $K = \pi/L \approx 0$ for the G-state of a bosonic system is used to determine the corresponding momenta through $k_i = q + K/2$

and $k_j = -q + K/2$ of two particles in superposition, each boson should either have π/d or $-\pi/d$ momentum and we find that these values differ from possible $k = \pm s\pi/L$ (with $s = 1, 2, 3, \dots$) considered in [4]. Similar estimates of k for a fermionic system using $q = q_o = \pi/d$ and K ranging between $\pi/L \approx 0$ and $K_F = N\pi/2L = \pi/2d$ render a band of allowed k ranging from $-q + K_F/2 = 3\pi/4d$ to $q + K_F/2 = 5\pi/4d$ (with least difference of $\pi/4L$) which do not match with $k = \pm 2s\pi/L$ considered in [6]. Note that these allowed k have been determined just for their comparison with those of [1-6], otherwise our analysis reveals that k_i and k_j do not represent good quantum numbers for the two particles in a state of wave superposition. Similarly, per particle G-state energy $\varepsilon_o = h^2/8md^2$ obtained by us for a bosonic system is three times higher than $(1/3)\varepsilon_o$ concluded in [4]. It may be noted that our results reveal a well defined picture of the G-state of 1-D systems. Accordingly, the system in its G-state is a close packed arrangement of the wave packets of its particles with $\langle x \rangle = d$, $q = q_o$ and $\Delta\phi = 2n\pi$ and this fact has been revealed for the first time. One may find a number of other points where our results differ from those of [1-6]. However, since an effective wave superposition of particles does not exist in higher energy states of $\lambda < d$, such particles can equally well be described by plane waves as considered in [1-6]. Evidently, such studies seem to provide reasonably accurate understanding of higher energy states (or high T phase) of the system and the present study, revealing the low energy states (or low T phase), can be identified to supplement the results of [1-6] in providing a complete understanding of a 1-D system. In view of an equivalence in the dynamical behavior of two HC particles interacting through V_{HC} and $A\delta(x)$, particularly, for low energy states of $\lambda/2 \geq \sigma$, it can be stated that our analysis also applies to particles of finite σ satisfying $\lambda/2 \geq \sigma$. Finally, as an important inference of this study, the true picture of the low energy states of our system can be revealed if its theory incorporates the impacts of the wave superposition and zero-point repulsion (*obvious effects of wave particle duality*), viz., $\langle x \rangle \geq \lambda/2$, $q \geq \pi/d$ and $\Delta\phi \geq 2n\pi$ on the relative configuration of two particles.

Acknowledgement : The author is thankful to Dr. P. K. Patra for useful discussion.

References

- [1] L. Tonks, Phys. Rev. **A50** 955 (1936).
- [2] M.D. Girardeau, J. Math. Phys. **1**, 516 (1960);
- [3] M. D. Girardeau, Phys. Rev. **139** B500 (1965).
- [4] E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
- [5] E.H. Lieb, Phys. Rev. **130**, 1616 (1963).
- [6] M. Flicker, and E.H. Lieb, Phys. Rev. **161**, 179 (1967).
- [7] C.N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
- [8] C.N. Yang, and C.P. Yang, J. Math. Phys. **10**, 1115 (1967).
- [9] M. Gaudin, Phys. Lett.**24A**, 55 (1967).
- [10] M. Gaudin, Phys. Rev. **A4**, 386 (1971).
- [11] J.B. McGuire, J. Math. Phys. **5**, 622 (1964).
- [12] J.B. McGuire, J. Math. Phys. **6**, 432 (1965).
- [13] J.B. McGuire, J. Math. Phys. **7**, 123 (1966).
- [14] A. Lenard, J. Math. Phys. **5** 930 (1964).
- [15] A. Lenard, J. Math. Phys. **7** 1268 (1966).
- [16] B. Sutherland, J. Math. Phys. **12**, 246 and 251 (1971).
- [17] B. Sutherland, Phys. Rev. **A5**, 1372 (1972).
- [18] H.G.Vaidya and C.A. Tracy, Phys. Rev. Lett. **42**, 3 (1979).
- [19] H.G.Vaidya and C.A. Tracy, Phys. Rev. Lett. **43**, 1540 (1979).
- [20] H.G.Vaidya and C.A. Tracy, J. Math. Phys. **20** 11 (1979).
- [21] M.Jimbo, T. Miwa, Y.Mori, and M. Sato, Phys. **D1** 80 (1980).
- [22] F. Calogero, J. Math. Phys. **10**, 2191 and 2197(1969), and **12**, 419 (1971).
- [23] H.B. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
- [24] V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983).
- [25] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).

- [26] D.C. Mattis, *The Many Body Problem: An Encyclopedia of Exactly Solved Models 1D* (World Scientific, 1994).
- [27] B. Sutherland, *Beautiful Models: 70 years of exactly solved quantum many-body problems*, (World Scientific, 2004).
- [28] W. Ketterle and N.J. van Druten, Phys. Rev. **A54** 656(1996).
- [29] M. T. Depue et al., Phys. Rev. Lett. **82** 2262 (1999).
- [30] A. Gorlitz, J.M. Vogels, A.E. Leanhardt, C. Raman, T.L. Gustavson, J.R. AboShaeer, A.P. Chikkatur, S. Gupta, S. Inouye, T.P. Rosenband, D.E. Pritchard, and W. Ketterle arXiv.org/Cond-mat/0104549 (2001).
- [31] A. Gorlitz et al Phys. Rev. Lett. **87** 130402 (2001)
- [32] A. Gorlitz et al., Nature **415** 39 (2002).
- [33] M. Greiner et al., Phys. Rev. Lett. **87**160405 (2001).
- [34] M. Greiner et al., Nature **415** 39 (2002).
- [35] S.Dettmer, D. Hellweg, P. Ryytty, J.J. Arlt, K. Sengstock, D.S. Petrov, G.V. Shlyapnikov, H. Kreutzmann, L. Santoz and M. Lewenstein, Phys. Rev. Lett. **87** 160406 (2001).
- [36] F. Schreck, et al., Phys. Rev. Lett. **87** 080403 (2001).
- [37] K. Bongs, et al Phys. Rev. **A63** 031602 (2001).
- [38] M. Olshanii, Phys. Rev. Lett. **81** 938 (1998).
- [39] D.S. Petrov, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. **85** 3745 (2000).
- [40] M. D. Girardeau, and E.M. Wright, Phys. Rev. Lett. **84** 5691 (2000).
- [41] M. D. Girardeau, and E.M. Wright, Phys. Rev. Lett. **84** 5239 (2000).
- [42] M. D. Girardeau, and E.M. Wright, and J.M. Triscari Phys. Rev. A **63** 033601 (2001).
- [43] V. Dunjko, V. Lorent and M. Olshanii, Phys. Rev. Lett. **86** 5413 (2001).
- [44] T. Papenbrock, arXiv.org/Cond-mat/0108474 (2001).
- [45] P. Ohberg and L. Santos, Phys. Rev. Lett. **89** 240402 (2002).
- [46] T. Bergman, M. Moore, and M. Olshanii, arXiv.org/Cond-mat/0210556 (2002).
- [47] M. Olshanii and V. Dunjko, . arXiv.org/Cond-mat/0210629 (2002).
- [48] G.E. Astrakharchik and S. Giorgini, arXiv.org/Cond-mat/0212512 (2002).

- [49] Kunal K. Das et al., Phys. Rev. **A65** 063603 (2002).
- [50] Kunal K. Das et al., Phys. Rev. Lett. **89** 170404 (2002).
- [51] M. D. Girardeau, Kunal K. Das and E.M. Wright, Phys. Rev. **A66** 023604 (2002).
- [52] D.M. Gangardt and G.V. Shlyapnikov, arXiv.org/Cond-mat/0207338 (2002).
- [53] K.V. Kheruntsyan, D.M. Gangardt, P.D. Drummond, and G.V. Shlyapnikov, arXiv.org/cond-mat/0212153 (2002)
- [54] P.J. Forrester, N.E. Frankel, T.M. Garoni and N.S. Witte, arXiv.org/Cond-mat/0211126 (2002).
- [55] K.V. Kheruntsyan, D.M. Gangardt, P.D. Drummond, and G.V. Shlyapnikov, Phys. Rev. Lett. **90** 010401 (2003).
- [56] M. D. Girardeau, arXiv.org/Cond-mat/0302213 (2003).
- [57] H.A. Bethe, Z. Physik **71**, 205 (1931).
- [58] (a) Y.S. Jain, *Wave mechanics of two hard core particles in one dimensional box*, arXiv.org/quant-ph/0603233 (2006),
(b) Y.S. Jain, Centl. Euro. J. Phys., 2(4), 709 (2004); The $\phi_k(x)^+ = \sqrt{2}|\sin(kx/2)|$ in Section 3.2 and in the discussion cited at Ref. [8] of this paper should be read as $\phi_k(x)^+ = \sqrt{2}\sin(|kx|/2)$.
- [59] K. Huang, *Statistical Mechanics*, Wiley Eastern Limited, New Delhi (1991).
- [60] L.I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw Hill, New York, 1968).
- [61] Y.S. Jain, *Untouched aspects of the wave mechanics of a particle in one dimensional box*, arXiv.org/quant-ph/0606009.
- [62] Y.S. Jain, J. Sc. Explor. **16**, 67(2002).
- [63] Y.S. Jain, *Basic problems of microscopic theories of many-body quantum systems*, arXiv.org/cond-mat/0208445.
- [64] Y.S. Jain, J. Sc. Explor. **16**, 77(2002); an improved updated version of this paper would be available on this arxiv (cond-mat/June 2006).
- [65] Y.S. Jain, J. Sc. Explor. **16**, 117(2002).
- [66] (a) Y.S. Jain, Ind. J. Phys. **79**, 1009 (2005).
Note: While the factor $|\sin(\mathbf{k}\cdot\mathbf{r})/2|$ in Eqn. 5 for Ψ^+ in this reference should be read as $\sin(|\mathbf{k}\cdot\mathbf{r}|/2)$, $E_g(T)/Nk_B T$ in Eqn. 25 be read as $E_g(T)/Nk_B$.
(b) Y.S. Jain, *Basic foundations of the microscopic theory of superconductivity*, arXiv.org/cond-mat/0603784.

Note: Although, $(\mathbf{q}, -\mathbf{q})$ bound pairs of electrons central to superconductivity as identified in this paper seem to be Cooper type but it may be emphasized that they are exactly not Cooper pairs; they have several similarities and dissimilarities with Cooper pairs.

- [67] W.P. Halperin, Rev. Mod. Phys. **58**, 533 (1986).
- [68] P. Kleban, Phys. Lett. **49A**, 19 (1974).
- [69] C.W. Woo in *The Physics of liquid and Solid Helium*, Part-I, Editors : K.H. Benneman and J.B. Ketterson, Wiley, New York (1976).
- [70] C. Kittel, *Introduction to Solid State Physics*, Fifth Edition, Wiley Eastern Limited, New Delhi (1983).
- [71] N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).
- [72] G. Taubes, Science, **269**, 152 (1995).
- [73] G.E. Uhlenbeck and L. Gropper, Phys. Rev. **41**, 79 (1932).
- [74] R.K. Pathria, *Statistical Mechanics*, Pergammon Press, Oxford (1976), Chapter 5.
- [75] J. Wilks, *The Properties of Liquid and Solid Helium*, Clarendon Press, Oxford (1967).
- [76] S.R. de Groot, G.J. Hooyman and C.A. Seldam, Proc. Royal Soc. London, **A203**, 266 (1950).
- [77] O. Carnal and J. Mlynek, Phys. Rev. Lett. **66**, 2689 (1991).
- [78] D.W. Keith, C.R. Ekstrom, Q.A. Turchette and D.E. Prochard, Phys. Rev. Lett. **66**, 2693 (1991).